to determine r, where $4^{804} \equiv r \pmod{100}$ and $r \in \mathbb{Z}$ with $0 \leq r < 100$.

(f) Determine the last two digits in the decimal representation of 3^{3356}.

(g) Determine the last two digits in the decimal representation of 7^{403}.

3.6 Review of Proof Methods

This section is different from others in the text. It is meant primarily as a review of the proof methods studied in Chapter 3. So the first part of the section will be a description of some of the main proof techniques introduced in Chapter 3. The most important part of this section is the set of exercises since these exercises will provide an opportunity to use the proof techniques that we have studied so far.

We will now give descriptions of three of the most common methods used to prove a conditional statement.

Direct Proof of a Conditional Statement $(P \to Q)$

- **When is it indicated**? This type of proof is often used when the hypothesis and the conclusion are both stated in a "positive" manner. That is, no negations are evident in the hypothesis and conclusion. That is, no negations are evident in the hypothesis and conclusion.

- **Description of the process**. Assume that P is true and use this to conclude that Q is true. That is, we use the forward-backward method and work forward from P and backward from Q.

- **Why the process makes sense**. We know that the conditional statement $P \to Q$ is automatically true when the hypothesis is false. Therefore, because our goal is to prove that $P \to Q$ is true, there is nothing to do in the case that P is false. Consequently, we may assume that P is true. Then, in order for $P \to Q$ to be true, the conclusion Q must also be true. (When P is true, but Q is false, $P \to Q$ is false.) Thus, we must use our assumption that P is true to show that Q is also true.

Proof of a Conditional Statement $(P \to Q)$ Using the Contrapositive

- **When is it indicated**? This type of proof is often used when both the hypothesis and the conclusion are stated in the form of negations. This often works well if the conclusion contains the operator "or"; that is, if the conclusion is in the form of a disjunction. In this case, the negation will be a conjunction.

- **Description of the process**. We prove the logically equivalent statement p $\neg Q \to \neg P$. The forward-backward method is used to prove $\neg Q \to \neg P$. That is, we work forward from $\neg Q$ and backward from $\neg P$.

- **Why the process makes sense**. When we prove $\neg Q \to \neg P$, we are also proving $P \to Q$ because these two statements are logically equivalent. When we prove the contrapositive of $P \to Q$, we are doing a direct proof of $\neg Q \to \neg P$. So we assume $\neg Q$ because, when doing a direct proof, we assume the hypothesis, and $\neg Q$ is the hypothesis of the contrapositive. We must show $\neg P$ because it is the conclusion of the contrapositive.

Proof of $(P \to Q)$ Using a Proof by Contradiction

- **When is it indicated**? This type of proof is often used when the conclusion is stated in the form of a negation, but the hypothesis is not. This often works well if the conclusion contains the operator "or"; that is, if the conclusion is in the form of a disjunction. In this case, the negation will be a conjunction.

- **Description of the process**. Assume P and $\neg Q$ and work forward from these two assumptions until a contradiction is obtained.

- **Why the process makes sense**. The statement $P \to Q$ is either true or false. In a proof by contradiction, we show that it is true by eliminating the only other possibility (that it is false). We show that $P \to Q$ cannot be false by assuming it is false and reaching a contradiction. Since we assume that $P \to Q$ is false, and the only way for a conditional statement to be false is for its hypothesis to be true and its conclusion to be false, we assume that P is true and that Q is false (or, equivalently, that $\neg Q$ is true). When we reach a contradiction, we know that our original assumption that $P \to Q$ is false is incorrect. Hence, $P \to Q$ cannot be false, and so it must be true.

Other Methods of Proof

The methods of proof that were just described are three of the most common types of proof. However, we have seen other methods of proof and these are described below.

Proofs that Use a Logical Equivalency

As was indicated in Section 3.2, we can sometimes use of a logical equivalency to help prove a statement. For example, in order to prove a statement of the form

$$P \to (Q \vee R), \tag{1}$$

it is sometimes possible to use the logical equivalency

$$[P \to (Q \vee R)] \equiv [(P \wedge \neg Q) \to R].$$

We would then prove the statement

$$(P \wedge \neg Q) \to R. \tag{2}$$

Most often, this would use a direct proof for statement (2) but other methods could also be used. Because of the logical equivalency, by proving statement (2), we have also proven the statement (1).

Proofs that Use Cases

When we are trying to prove a proposition or a theorem, we often run into the problem that there does not seem to be enough information to proceed. In this situation, we will sometimes use cases to provide additional assumptions for the forward process of the proof. When this is done, the original proposition is divided into a number of separate cases that are proven independently of each other. The cases must be chosen so that they exhaust all possibilities for the hypothesis of the original proposition. This method of case analysis is justified by the logical equivalency

$$(P \vee Q) \to R \equiv (P \to R) \wedge (Q \to R),$$

which was established in Preview Activity 1 in Section 3.4.

Constructive Proof

This is a technique that is often used to prove a so-called **existence theorem.** The objective of an existence theorem is to prove that a certain mathematical object exists. That is, the goal is usually to prove a statement of the form

$$\text{There exists an } x \text{ such that } P(x).$$

For a constructive proof of such a proposition, we actually name, describe, or explain how to construct some object in the universe that makes $P(x)$ true.

Nonconstructive Proof

This is another type of proof that is often used to prove an existence theorem is the so-called **nonconstructive proof.** For this type of proof, we make an argument that an object in the universal set that makes $P(x)$ true must exist but we never construct or name the object that makes $P(x)$ true.

Exercises for Section 3.6

1. Let h and k be real numbers and let r be a positive number. The equation for a circle whose center is at the point (h, k) and whose radius is r is

$$(x-h)^2 + (y-k)^2 = r^2.$$

We also know that if a and b are real numbers, then

- The point (a, b) is inside the circle if $(a-h)^2 + (b-k)^2 < r^2$.
- The point (a, b) is on the circle if $(a-h)^2 + (b-k)^2 = r^2$.
- The point (a, b) is outside the circle if $(a-h)^2 + (b-k)^2 > r^2$.

Prove that all points on or inside the circle whose equation is $(x-1)^2 + (y-2)^2 = 4$ are inside the circle whose equation is $x^2 + y^2 = 26$.

2. Let r be a positive real number. The equation for a circle of radius r whose center is the origin is $x^2 + y^2 = r^2$.

(**a**) Use implicit differentiation to determine $\dfrac{dy}{dx}$.